INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 4676

Décembre 2002

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4676--FR+ENG

$\mathcal{T}$ube derivative for non-cylindrical problems Application to Shape-Newton Method

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Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Opale

Rapport de recherche n° 4676 — Décembre 2002 — 26 pages

Résumé : The purpose of this preport is to give the expression of the shape derivative of functionals related to moving domains evolution problems. The main step is to define pertubations of a given non-cylindrical domain [also called tube] using transverse vector fields. Under a smoothness assumption, a tube can be generated by a suitable vector field starting from its initial domain. This relationship tube-vector field allows us to define the field derivative and to recognize the term which constitutes the shape derivative for a class of non-cylindrical functionals. We conclude the paper, by applying the obtained result to compute the second order optimality condition for a shape minimization problem.

Mots-clés : Non-cylindrical shape functionals, Eulerian derivatives, transverse vector fields, shape gradient and hessian

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Dérivée de forme pour des problèmes non-cylindriques

Abstract: Le but de ce rapport est de donner l'expression de la derivée de forme de fonctionnelles associées à des problmes d'évolution non-cylindrique. Le point clé de ce travail réside dans la définition des perturbations d'un domaine non-cylindrique [appelé aussi tube] en utilisant des champs de vecteurs transverses. Sous des hypothèses de régularités, un tube peut être généré par un champ de vecteur convenable connaissant son domaine initial. Cette relation tube-champ de vecteur nous permet de définir une dérivée par rapport aux champs de vitesse et d'en déduire l'expression de la dérivée par rapport à la forme. On analyse la dérivation de fonctionnelle de tube associées à des problèmes dynamiques non cylindriques. On considère le cas particulier de l'èvolution d'une fonctionnelle de domaine et l'application aux méthode d'ordre 2 de type Newton; conclut ce rapport, en donnant la condition nécessaire s d'optimalité du second ordre(faisaqnt intervenir le hessien de forme) pour un tel problème de minimisation.

Key-words: Fonctionnelles de forme non-cylindrique, dérivation par rapport aux champs de vitesse, champ transverse, dérivation par rapport à la forme

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1 Introduction

Dynamical shape control problems for systems modelized with dynamical non-cylindrical Partial Differential Equations (PDE) are encountered in fluid-structure issues, free boundary problems, etc...
The cost functional involved in such problems is expressed in terms of integrals over the non-cylindrical evolution domain and/or its lateral boundary. Following [] this non-cylindrical evolution domain will be called a tube and is of the following form :

$$Q = \bigcup_{0<t<\tau} (\{t\} \times \Omega_t); \quad \text{At } \ t = 0, \ \Omega_0 = \Omega.$$

The initial geometry Ω is called the base of the tube. Non smooth tubes are described in [11] with help of the time convection of the base by non lipschitzian vector field V. If the lateral boundary Σ of the tube Q is smooth enough to guarantee the existence of the outward normal field $\vec{\nu}$ to Σ, there exists smooth non-autonomous vector fields V such that

$$T_t(V)\Omega = \Omega_t \ \subset R^N \ \forall t \in [0, \tau) \tag{1}$$

where $T(V)$ is the flow associated to V. More precisely, the *intrinsic* outward normal field $\vec{\nu} \in R^{N+1}$ defined on the lateral boundary Σ of Q, can be written as (cf.[9])

$$\vec{\nu}(t) = \frac{1}{\sqrt{1+v_\nu^2}}(-v_\nu(t), \vec{n}_{\Omega_t})$$

where $\vec{n}_{\Omega_t} \in R^N$ is the "horizontal" outward normal field to Ω_t. Then any sufficiently smooth non-autonomous vector field V such that

$$V(t).\vec{n}_{\Omega_t} = v_\nu(t) \quad \text{on} \ \ \Sigma \tag{2}$$

builds the tube Q (or equivalently : satisfies (1)).
Conversely, to any non-autonomous vector field V, one can associate, in the time interval $[0, \tau]$, a tube $Q(V)$ (also denoted Q_V) by setting $\Omega_t = T_t(V)(\Omega)$.

Obviously (2) is then satisfied on the lateral boundary $\Sigma(V)$ of $Q(V)$.
The functionals considered may depend not only on the tube containing the evolution takes place but also on a field that builds this tube which verifies some extra physical condition (cf. [6]).In that specific situation, the functional is no more a tube functional as it depends on the tube but also on a specific vector field V. For that reason we consider now as independant variable in the problem, not the tube itself but the vector field V whose flow mapping builts the tube starting from a given base Ω. We designate by Q_V the tube built by an admissible vector field V and we consider functionals $\mathbf{j(V)}$ in the form

$$\mathbf{j(V) = J(V, Q_V)}.$$

When J depends exclusively on Q_V we get :

$$\mathbf{j(V+W) = j(V)}, \ \forall \ \mathbf{W} \ \mathbf{s.t.} \ < \mathbf{W}, \ \mathbf{n}_{\mathbf{\Omega_t(V)}} >= \mathbf{0} \ \text{ on } \mathbf{\Sigma_V},$$

this means that $\mathbf{j}$ depends only on the shape of the tube (it is, thus, called a tube functional). The dependence of J on a field V comes, generally, from boundary conditions associated to the state equation.
For a bounded tube Q, there exists a bounded open set D (called a hold-all) such that Q and its perturbations remain in $(0, \tau) \times D$. That cylindrical tube will be invariant under all the transformations we shall consider so that the vector field V as well as the preturbations fields W will satisfy the condition :

$$< W(t), \vec{n}_D >= 0 \quad \text{on} \quad (0, \tau) \times \partial D \tag{3}$$

where $\vec{n}_D$ is the outward normal field to D, cf. for example [8] or [9].
The aim is to characterize the derivative with respect to the field of the functional $\mathbf{j}'(\mathbf{V}, \mathbf{W})$. In a first step we obatain a general expression in terms of the *transverse* field Z. We study the problem whose Z is solution and we explicit the gradient with use of the transposed equation solution Λ..
We give sufficient conditions under which $\mathbf{j}$ is Gâteaux differentiable at a field V. For an optimization problem, we prove that if the state depends on the field only by the boundary conditions, the field gradient $G(V)$ is supported by Σ_V. But if it depends only on the shape of the tube, $G(V)$ has the following form

$$G(V)(t) = \gamma^*_{\Gamma_t(V)}(g(V)(t) n_{\Omega_t(V)})$$

where $\Gamma_t(V) = \partial(\Omega_t(V))$ and $\gamma^*_{\Gamma_t(V)}$ is the adjoint of the trace operator on $\Gamma_t(V)$, $g(V)(t) \in [\mathcal{D}^{k-1}(\Gamma_t(V))]'$. For these results one has to assume Ω_t to be $\mathcal{C}^k$, $\forall t \in [0, \tau]$.

2 Basic tools

2.1 Transformations-Velocity method

First, let us recall some basic results dealing with velocity flows. Let $\tau > 0$, D a convex, smooth and bounded open set in R^N. Consider a vector field

$$V : [0,\tau] \times \overline{D} \longrightarrow R^N, (t,x) \longmapsto V(t)(x) \stackrel{def}{=} V(t,x) \text{ such that}$$

$$V(t).n_D = 0 \text{ on } \partial D, \forall t \in [0,\tau]. \tag{4}$$

Moreover, we assume

$$(V) \left\{ \begin{array}{c} \forall x \in \overline{D}, V(.,x) \in C([0,\tau]; R^N) \\ \\ \exists c > 0, \forall x, y \in D, \| V(.,x) - V(.,y) \|_{C([0,\tau];R^N)} \leq c|x-y| \end{array} \right.$$

where $V(.,x)$ is the function $t \longmapsto V(t,x)$. For any $X \in \overline{D}$, associate the solution $x_V(.,X)$ of the ordinary differential equation

$$\left\{ \begin{array}{l} \dfrac{dx}{dt}(t) = V(t,x(t)), t \in (0,\tau) \\ x(0) = X \end{array} \right.$$

For any $t \in [0,\tau]$, we have a transformation

$$T_t(V) : \overline{D} \longrightarrow \overline{D};\ X \longmapsto T_t(V)(X) \stackrel{def}{=} x_V(t,X).$$

The mapping $(t,X) \longmapsto T_t(V)(X)$ is denoted $T(V)$ or T if no confusion is possible. Under assumption (V), the map T has the following properties :

$$\left\{ \begin{array}{ll} (T_1) & \begin{array}{ll} \forall X \in \overline{D}, & T(.,X) \in C^1([0,\tau]; R^N) \text{ and } \exists c > 0, \\ \forall X, Y \in \overline{D}, & ||T(.,X) - T(.,Y)_{C^1([0,\tau];R^N)}|| \leq c\ |X-Y| \end{array} \\ \\ (T_2) & \forall t \in [0,\tau], \quad X \longmapsto T_t(X) : \overline{D} \longrightarrow \overline{D} \text{ is bijective} \\ \\ (T_3) & \begin{array}{ll} \forall x \in \overline{D}, & T^{-1}(.,x) \in C([0,\tau]; R^N) \text{ and } \exists c > 0, \\ \forall x, y \in \overline{D}, & ||T^{-1}(.,x) - T^{-1}(.,y)_{C([0,\tau];R^N)}|| \leq c\,|x-y| \end{array} \end{array} \right.$$

Introduce the Banach space, $k \in N$

$$\mathcal{V}_o^k(D) = \{v \in C^k(\overline{D}, R^N)) \mid v.n_D = 0 \text{ on } \partial D\}$$

and the following notations :

$$c(F) \stackrel{def}{=} \sup_{y \neq x} \frac{|F(y) - F(x)|}{|y-x|};\ c_k(F) \stackrel{def}{=} \sum_{|\alpha|=k} c(\partial^\alpha F) \text{ for } k \geq 1.$$

The regularity result stated below for the flow associated to a given vector field V is developed in [3].

Proposition 1
For all $V \in C([0,\tau], \mathcal{V}_o^k(D))$, $k \in N$, such that

$$c_k(V(t)) \leq c \text{ (for a constant } c > 0 \text{ independent of } t), \tag{5}$$

it is associated a unique map

$$T(V) \in C^1([0,\tau], C^k(\overline{D}, R^N)) \cap C([0,\tau], W^{k+1,\infty}(D, R^N)).$$

Moreover the mapping $t \longrightarrow V(t) \circ T_t$ is in $L^\infty(0,\tau; W^{k+1,\infty}(D, R^N))$.

The transformation $T_t(V)$ is one-to-one and maps $\overline{D}$ into $\overline{D}$. For each $t \in [0,\tau]$, one can consider the transformation T_t^{-1} and notice that it is the flow, at $s = t$, of the vector field V_t defined by $V_t(s) = -V(t-s)$ (cf. [8], [3]).Using this argument we can prove that $T^{-1} \in C([0,\tau], C^k(\overline{D}, R^N))$ and with the implicit function theorem one can prove the following regularity result.

Proposition 2
The mapping $[0,\tau] \longrightarrow \mathcal{C}^k(\overline{D}, R^N), t \longmapsto T_t^{-1}$ is continuously differentiable.

Proof. Let $t \in (0,\tau)$. The map $T_t \in \mathcal{C}^k(\overline{D}, R^N)$. Consider the mapping

$$\Phi : [0,\tau] \times \mathcal{C}^{k-1}(\overline{D}, R^N) \longrightarrow \mathcal{C}^{k-1}(\overline{D}, R^N)$$

defined by

$$\Phi(t,S) = T(V)(t,S) - Id$$

It is clear that $\Phi(t, T_t(V)^{-1}) = 0$. Moreover Φ is continuously differentiable. The partial derivatives are :

$$\partial_t \Phi(t,S) = V(t, T_t(V) \circ S); \;\; \partial_S \Phi(t,S)\xi = (DT_t) \circ S.\xi$$

It is clear that $\partial_S \Phi(t,S)$ is in aut $(\mathcal{C}^{k-1}(\overline{D}, R^N))$. We deduce from the implicit function theorem that $T^{-1} \in \mathcal{C}^1([0,\tau]; \mathcal{C}^{k-1}(\overline{D}, R^N))$. Then the mapping $t \longrightarrow DT_t^{-1} = (DT_t)^{-1} \circ T_t^{-1}$ is also in $\mathcal{C}^1([0,\tau]; \mathcal{C}^{k-1}(\overline{D}, R^N))$. Therefore the announced regularity is proved. $\square$

2.2 Tube perturbation

From now on we assume $k \geq 1$. A perturbation of a tube Q_V in a direction W can be obtained by considering transverse transformations (which are transformations acting on domains built at the same time).

Let $\Omega \subset\subset D$ of class C^k be given, for any sufficiently small positive parameter s, we consider the moving domain $Q_{(V+sW)}$ as the perturbation of the tube Q_V in the direction of the field W. It is composed of the sets

$$\Omega_t(V+sW) = T_t(V+sW)(\Omega),\ \forall t \in [0,\tau].$$

A transverse transformation is a function which maps $\Omega_t(V)$ onto $\Omega_t(V+sW)$ for any $t \in [0,\tau]$ (and $\overline{D}$ onto $\overline{D}$). A quite natural one is

$$\mathcal{T}_s^t = T_t(V+sW) \circ T_t(V)^{-1}.$$

If the regularity assumptions $(T_1)-(T_3)$ are satisfied by the mapping $(s,x) \longmapsto \mathcal{T}_s^t(x)$ for any $t \in [0,\tau]$, this transformation can be considered as the flow of the vector field [see for instance [1]]

$$\mathcal{Z}^t(s,.) = \left(\frac{\partial}{\partial s}\mathcal{T}_s^t\right) \circ \mathcal{T}_s^t(\ .\ )^{-1} = [\partial_s T_t(V+sW)] \circ T_t(V+sW)^{-1}. \qquad (6)$$

Lemme 1
Let I_0 be a neighborhood of zero; V and W being in $\mathcal{C}([0,\tau];\mathcal{V}_o^k(D))$ satisfying condition (5). The mapping

$$I_0 \longrightarrow \mathcal{C}([0,\tau];C^{k-1}(\overline{D},R^N))$$
$$s \longrightarrow T(V+sW)$$

is continuously differentiable and $\partial_s(T_t(V+sW))$ satisfies for any $t \in [0,\tau]$,

$$\partial_s[T_t(V+sW)] = \int_0^t D(V+sW)(\mu, T_\mu(V+sW))\partial_s[T_\mu(V+sW)]\, d\mu$$
$$+ \int_0^t W(\mu, T_\mu(V+sW))\, d\mu. \qquad (7)$$

Proof.

$$T_t(V+sW) - T_t(V+s_oW)$$
$$= \int_0^t (V+sW)(\mu, T_\mu(V+sW)) - (V+s_oW)(\mu, T_\mu(V+s_oW))\, d\mu.$$

$$||T_t(V+sW) - T_t(V+s_oW)_{C^{k-1}(\overline{D})}|| \le |s-s_o \int_0^t ||W(\mu)||_{C^{k-1}(\overline{D})}\, d\mu$$
$$+ \max_{t\in[0,\tau]} ||DV(t) + s_oDW(t)||_{C^{k-1}(\overline{D})} \int_0^t ||T_\mu(V+sW) - T_\mu(V+s_oW)||_{C^{k-1}(\overline{D})}\, d\mu$$

Applying the Gronwall inequality, it comes

$$||T_t(V+sW)-T_t(V+s_oW)||_{C^{k-1}(\overline{D})} \le \tau|s-s_o|||W_{C([0,\tau];C^{k-1}(\overline{D}))}||$$
$$+\tau|s-s_o|\ ||W_{C([0,\tau];C^{k-1}(\overline{D}))}|| \int_0^t \exp(t-\mu)\,d\mu$$

Thus for any $t \in [0,\tau]$:

$$||T_t(V+sW)-T_t(V+s_oW)||_{C^{k-1}(\overline{D})} \le \tau|s-s_o|\ ||W||_{C([0,\tau];C^{k-1}(\overline{D}))}\ e^t.$$

This proves that the considered map is in $W^{1,\infty}(I_0;C([0,\tau];C^{k-1}(\overline{D};R^N)))$ and also gives the following uniform boundedness (with respect to s) :

$$\frac{1}{|s-s_o}|\ ||(V+sW)(\mu,T_\mu(V+sW))-(V+s_oW)(\mu,T_\mu(V+s_oW))||_{C^{k-1}(\overline{D})}$$
$$\le ||W(\mu)||_{C^{k-1}(\overline{D})}+\tau\exp\tau \max_{t\in[0,\tau]}||DV(t)+s_oDW(t)||_{C^{k-1}(\overline{D})}\,||W(\mu)||_{C^{k-1}(\overline{D})}$$

Then according to the Lesbegue theorem, the derivative exists everywhere in I_0 and satisfies (7). It has the following expression :

$$\partial_s[T_t(V+sW)] = \int_0^t \exp\{\int_\xi^t D(V+sW)(\mu,T_\mu(V+sW))\,d\mu\}W(\xi,T_\xi(V+sW))\,d\xi.$$
$$= \int_0^t DT_t(V+sW).[DT_\xi(V+sW)]^{-1}W(\xi,T_\xi(V+sW))\,d\xi.$$

It is clear that this expression is continuous in I_0.

$\square$

Let $\mathcal{S}^t(s)=\partial_s[T_t(V+sW)]$. Then $\mathcal{Z}^t(s,x)=\mathcal{S}^t(s)\circ T_t(V+sW)^{-1}$. In the next section, it will be shown that the field derivatives of non-cylindrical functionals are expressed in terms of the transverse vector field $\mathbf{Z}(t,x)=\mathcal{Z}^t(0,x)$. Therefore, one has to know more about this vector field. In particular, it will be shown that $\mathbf{Z}$ can be characterized as the unique solution of

$$\partial_t\mathbf{Z}+[\mathbf{Z},V] = W \text{ in } (0,\tau)\times D \tag{8}$$
$$\mathbf{Z}(0,.) = 0 \text{ in } D \tag{9}$$

where [,] denotes the Lie Brackets. For that purpose let us consider the vector field S, $S(t,.)\stackrel{def}{=}\mathcal{S}^t(0,.)=\mathbf{Z}(t)\circ T_t(V)$.

Lemme 2
The function S is the unique vector field, in $\mathcal{C}^1([0,\tau];C^{k-1}(\overline{D},R^N))$, satisfying

$$S(t)=\int_0^t W(\mu,T_\mu(V))d\mu+\int_0^t DV(\mu,T_\mu(V))S(\mu)\,d\mu. \tag{10}$$

Proof. Let $\mathcal{F}$ be the mapping defined by

$$\mathcal{F} : [0,\tau] \times C^{k-1}(\overline{D}, R^N) \to C^{k-1}(\overline{D}, R^N)$$

$$(t,\varphi) \to DV(t, T_t(V))\varphi + W(t, T_t(V)).$$

For any $t \in [0,\tau]$ and any $\varphi \in C^{k-1}(\overline{D})$, $\mathcal{F}(t)$ is affine and $\mathcal{F}(.,\varphi)$ is continuous. So the existence and uniqueness of (10) are given by the Cauchy-Lipschitz theorem. Moreover the solution has the following expression

$$S(t) = \int_0^t \exp\{\int_s^t DV(\mu, T_\mu(V))\, d\mu\} W(s, T_s(V))\, ds, \quad \forall t \in [0,\tau].$$

$$= \int_0^t DT_t(V).[DT_s(V)]^{-1} W(s, T_s(V))\, ds$$

□

Lemme 3
If $\psi \in C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$ is such that

$$\partial_t \psi + D\psi.V \ \in C([0,\tau], C^{k-1}(\overline{D}, R^N))$$

and satisfies (8)-(9), then $\psi \circ T(V)$ belongs to $C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$ and satisfies (10). Conversely, if $\varphi \in C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$ is solution of (10), then $\varphi \circ T(V)^{-1} \in C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$ such that

$$\partial_t[\varphi \circ T(V)^{-1}] + D[\varphi \circ T(V)^{-1}].V \ \text{ is in } \ C([0,\tau], C^{k-1}(\overline{D}, R^N))$$

and satisfies (8)-(9).

Proof. If ψ belongs to $C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$, then it is the same for the mapping $t \longmapsto \psi(t) \circ T_t$ and we have :

$$\partial_t(\psi(t, T_t)) = [DV(t)] \circ T_t.\, \psi(t, T_t) + W(t) \circ T_t.$$

Thus $\psi \circ T(V)$ satisfies (10). Conversely, since $\varphi \in C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$ is solution of (10) and T^{-1} is in $C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$, it comes that $\varphi \circ T(V)^{-1}$ is in $C^1([0,\tau], C^{k-1}(\overline{D}, R^N))$ and we have :

$$\begin{aligned}
&\partial_t(\varphi(t, T_t^{-1})) = [\partial_t \varphi] \circ T_t^{-1} + [D\varphi] \circ T_t^{-1}.\partial_t(T_t^{-1}) \\
&= \{[DV] \circ T_t.\varphi + W \circ T_t\} \circ T_t^{-1} - (D\varphi) \circ T_t^{-1} D(T_t^{-1}).V(t) \\
&= DV.(\varphi \circ T_t^{-1}) + W - D(\varphi \circ T_t^{-1}).V(t)
\end{aligned}$$

which concludes the proof.

□

Theorem 1
The field $\mathbf{Z}$ is the unique vector field in $C^1([0,\tau],C^{k-1}(\overline{D},R^N))$, such that $\partial_t \mathbf{Z} + D\mathbf{Z}.V \in C([0,\tau],C^{k-1}(\overline{D},R^N))$, solution of problem (8)-(9).

Proof. Consider the solution S of (10), compose by $T(V)^{-1}$ and make use of the lemma 3. □

Remark 1
Taking into account the characterization of $\mathbf{Z}$, we deduce that it has the following expression :

$$\mathbf{Z}(t) = \{\int_0^t DT_t(V).[DT_s(V)]^{-1} W(s,T_s(V))\, ds\} \circ T_t(V)^{-1}. \tag{11}$$

By the way a useful property of $\mathbf{Z}$ is that if V and W are of free divergence then $\mathbf{Z}$ too. Indeed

Proposition 3
Let V and W in $C([0,\tau],\mathcal{V}_o^k(D)), k \geq 1$, such that (5) holds. Assume

$$\mathrm{div} V = \mathrm{div} W = 0 \ \text{ in } D.$$

Then the field $\mathbf{Z}$ is of divergence free :

$$\mathrm{div}\mathbf{Z} = 0 \ \text{ in D }.$$

Proof. Let $f \in \mathcal{D}(D)$. The transformations $T_t(V+sW)$ and $T_t(V)^{-1}$ maps D onto D. Then,

$$\int_D f \circ \mathcal{T}_s^t \, dx = \int_D f \, dx.$$

Indeed since V and W are of free divergence, we have

$$\int_D f\, dx = \int_{T_t(V+sW)(T_t(V)^{-1}(D))} f\, dx = \int_{T_t(V)^{-1}(D)} f \circ T_t(V+sW)\, dx = \int_D f \circ \mathcal{T}_s^t\, dx.$$

From this we deduce that

$$\frac{d}{ds}\left(\int_D f \circ \mathcal{T}_s^t\, dx\right) = 0$$

The mapping $s \longmapsto \mathcal{T}_s^t$ is in $C^1(I_0; C^{k-1}(\overline{D},R^N))$, thus

$$\int_D \nabla f \,.\, \mathbf{Z}\, dx = 0, \forall f \in \mathcal{D}(D).$$

or equivalently $\mathrm{div}\,\mathbf{Z} = 0$ in $\mathcal{D}'(D)$. □

2.3 Explicit expression of $\mathbf{Z}(t).n_{\Omega_t(V)}$

In the sequel, we consider a set Ω of class $\mathcal{C}^k$ (k is kept greater than 1). As we will see in the applications, the expression of the Eulerian derivative depends on $\mathbf{Z}$. Hence, it seems to be necessary to introduce two adjoint states. One associated to the state equation and the other to the field $\mathbf{Z}$. For a functional defined on a tube Q_V (with the initial domain Ω), this unusual situation might be avoided by considering the function $\mathbf{z}$ defined by

$$\mathbf{z}(t) = (\mathbf{Z}(t).n_t) \circ T_t(V) \;\text{ on }\; (0,\tau) \times \Gamma \;\; (\Gamma = \partial\Omega).$$

Lemme 4
The mapping $t \longmapsto T_t(V)$ is in $C^1([0,\tau]; C^k(\overline{D}, R^N))$. Therefore

$$t \longmapsto n_t \circ T_t = \frac{{}^*(DT_t)^{-1}\, n}{||\, {}^*(DT_t)^{-1}\, n||} \;\text{ is in }\; C^1([0,\tau]; C^{k-1}(\Gamma))$$

n and n_t are the outward normal fields respectively to Ω and $\Omega_t(V)$, on Γ and Γ_t. Its derivative is given by :

$$\partial_t(n_t \circ T_t) = < DV.n_t, n_t > \circ T_t\; n_t \circ T_t - {}^*DV \circ T_t\; n_t \circ T_t.$$

Proposition 4
The function $\mathbf{z} \in C^1([0,\tau]; C^{k-1}(\Gamma))$ is the unique solution of

$$\partial_t \mathbf{z}(t) \,-\, \alpha(t) \circ T_t(V)\, \mathbf{z}(t) = \beta(t) \circ T_t(V) \;\text{ on }\; (0,\tau) \times \Gamma \tag{12}$$
$$\mathbf{z}(0) = 0 \;\text{ on }\; \Gamma \tag{13}$$

where $\alpha(t) = < DV.n_t, n_t >$, $\beta(t) = W(t).n_t$.

Proof. The mapping $t \longmapsto \mathbf{Z}(t, T_t) = S(t)$ is differentiable. Thus

$$\begin{aligned}
&\partial_t\left[\mathbf{Z}(t,T_t).\, n_t \circ T_t\right] = \partial_t(\mathbf{Z} \circ T_t)\,.\, n_t \circ T_t + \mathbf{Z}(t,T_t).\partial_t(n_t \circ T_t)\\
&= < (W(t) + DV(t).Z(t)) \circ T_t\,,\, n_t \circ T_t > \; + < DV.\, n_t, n_t > \; \circ T_t\, (\mathbf{Z}(t).n_t) \circ T_t\\
&- < (DV.\mathbf{Z}) \circ T_t\,,\, n_t \circ T_t >\\
&= (W(t).\, n_t) \circ T_t + < DV.n_t, n_t > \circ T_t\; (\mathbf{Z}(t).\, n_t) \circ T_t.
\end{aligned}$$

Eventually the desired result is obtained.

□

Remark 2
Expression of $\mathbf{z}$ in terms of the data :
From (12) we deduce that

$$\partial_t[\mathbf{z}\,\exp(\int_0^t \alpha(s)\circ T_s\,ds)] = [\alpha(t)\circ T_t\,\mathbf{z} + \partial_t\mathbf{z}\,]\exp(\int_0^t \alpha(s)\circ T_s\,ds)$$
$$= \beta(t)\circ T_t\exp(\int_0^t \alpha(s)\circ T_s\,ds).$$

Hence $\mathbf{z}$ can be expressed as follows

$$\mathbf{z}(t) \;=\; \int_0^t \beta(s)\circ T_s(V)\;\exp(-\int_s^t \alpha(r)\circ T_r(V)\,dr)\,ds. \tag{14}$$

$$= \int_0^t [W(s).n_s]\circ T_s(V)\;\exp(-\int_s^t < DV(r)n_r\,,\,n_r > \circ T_r(V)\,dr)\,ds.$$

2.4 Adjoint problem associated to Z

As shown in the proof of Lemma 3 the solution of (8)-(9) is obtained via a change of variable. Then if $H(D)$ is a Banach space of functions defined on D, stable by multiplication by functions in $C^{k-1}(\overline{D})$, the same process generates the solution of the adjoint problem associated to $\mathbf{Z}$.

Theorem 2
Let $F\in L^2((0,\tau);H(D))$. There exists a unique $\Lambda\in C([0,\tau];H(D))$ such that $\partial_t\Lambda+D\Lambda.V\in L^2((0,\tau);H(D))$ solution of

$$-\,\partial_t\Lambda - D\Lambda.V - {}^*DV.\Lambda - (\mathrm{div}V)\,\Lambda \;=\; F \tag{15}$$
$$\Lambda(\tau) \;=\; 0. \tag{16}$$

Proof. Consider $\theta\in C^1([0,\tau];H(D))$ the unique solution of the backward problem

$$-\partial_t\theta - [\,{}^*(DV)\circ T_t + (\mathrm{div}V)\circ T_t\,]\;\theta \;=\; F\circ T_t$$
$$\theta(\tau) \;=\; 0.$$

Applying $\exp\int_0^t [({}^*DV(s))\circ T_s + (\mathrm{div}V(s))\circ T_s\,\mathbf{I}]\;ds$, we get

$$-\partial_t\left[\exp\{\int_0^t [\,{}^*(DV(s))\circ T_s + (\mathrm{div}V(s))\circ T_s\,\mathbf{I}]\;ds\}\theta(t)\right] =$$
$$\exp\{\int_0^t [\,{}^*(DV(s))\circ T_s + (\mathrm{div}V(s))\circ T_s\;\mathbf{I}]\;ds\}\;F\circ T_t.$$

By integration we deduce an explicit expression of θ :

$$\theta(t) = \int_t^\tau \exp\{-\int_s^t [\,{}^*DV(\xi) \circ T_\xi + (\mathrm{div}V(\xi)) \circ T_\xi \,\mathbf{I}\,]\, d\xi\}\; F(s) \circ T_s \;\; ds.$$

$$= \int_t^\tau {}^*(DT_t)^{-1}\, {}^*(DT_s) F(s) \circ T_s \gamma(s)\gamma(t)^{-1}\, ds$$

Then taking for $\Lambda = \theta \circ T_t^{-1}$, it is easy to see that Λ is the unique solution of (15)-(16) which, for a suitable right-hand term, will represent the adjoint problem associated to $\mathbf{Z}$ associated to a given cost functional.

□

2.5 A right-hand term supported by $\Sigma(V)$

Let $f \in L^2(\Sigma(V))$ and assume that the mapping $t \longmapsto \gamma_{\Gamma_t}^*(f(t)n_t)$ belongs to $L^2((0,\tau), H(D))$. We proved in Theorem 2 the existence of a unique $\Lambda \in C([0,\tau], H(D))$ such that

$$\begin{aligned} -\,\partial_t\Lambda - D\Lambda.V - {}^*DV.\Lambda - \mathrm{div}V\;\Lambda &= \gamma_{\Gamma_t}^*(f(t)n_t) \\ \Lambda(\tau) &= 0. \end{aligned} \tag{17}$$

We shall prove that the solution Λ is, in fact, supported by the lateral boundary $\Sigma(V)$ since the right-hand term in this problem is itself supported by $\Sigma(V)$ and there is no diffusion term.

Lemme 5
Let $f \in L^2(0,\tau; L^2(\Gamma_t))$. There exists a unique solution in $C([0,\tau]; L^2(\Gamma_t))$ such that $\partial_t\lambda + \nabla_{\Gamma_t}\lambda.V \in L^2(0,\tau; L^2(\Gamma_t))$ of the following problem

$$\left\{ \begin{array}{lcl} \partial_t\lambda(t) + \nabla_{\Gamma_t}\lambda.V + \lambda\,\mathrm{div}V &=& f(t) \;\text{ on }\; \cup_t(\{t\}\times\Gamma_t) \\ \lambda(\tau) &=& 0 \;\text{ on }\; \Gamma_\tau \end{array} \right.$$

Proof. Notice that

$$[\partial_t\lambda(t) + \nabla_{\Gamma_t}\lambda.V]_{|\Gamma_t} \circ T_t = \partial_t(\lambda \circ T_t)_{|\Gamma}$$

and consider $\mu \in C([0,\tau]; L^2(\Gamma))$ the unique solution of

$$\begin{aligned} &\partial_t\mu + (\mathrm{div}V) \circ T_t \;\; \mu = f(t) \circ T_t \;\text{ on }\; (0,\tau)\times\Gamma \\ &\mu(\tau) = 0 \;\text{ on }\; \Gamma. \end{aligned}$$

which can be expressed as :

$$\begin{aligned} \mu(t) &= -\int_t^\tau \exp\{\int_t^s (\mathrm{div}V) \circ T_r\, dr\}\, f(s) \circ T_s\, ds \\ &= -\int_t^\tau \mathrm{det}DT_s.(\det DT_t)^{-1}\, f(s) \circ T_s\, ds \end{aligned}$$

Then λ, defined by $\lambda(t) = \mu(t) \circ T_t^{-1}$, is solution of the considered problem. Uniqueness is obvious.

$\square$

Theorem 3
Let $f \in L^2(0, \tau; L^2(\Gamma_t))$. The solution Λ of (17) is supported by $\Sigma(V)$. Precisely

$$\Lambda(t) = -\gamma_{\Gamma_t}^*(\lambda(t) n_t), \; t \in (0, \tau). \tag{18}$$

where λ is defined in Lemma 5.

For further properties of $p(.)$ see [2] or [5].

Proof. Set $\mathbf{X}(t) = -\gamma_{\Gamma_t}^*(\lambda(t) n_t)\,(\in H^{-1}(D, R^N))$. We should identify the distribution

$$-\partial_t \mathbf{X} - D\mathbf{X}.V - {}^*DV.\mathbf{X} - (\mathrm{div} V)\,\mathbf{X}.$$

For that let $\varphi \in \mathcal{D}((0,\tau) \times D)$, thus

$$\begin{aligned}
&< -\partial_t \mathbf{X} - D\mathbf{X}.V - {}^*DV.\mathbf{X} - (\mathrm{div} V)\,\mathbf{X}, \varphi >_{\mathcal{D}'((0,\tau)\times D), \mathcal{D}((0,\tau)\times D)} \\
&= -\int_0^\tau \int_{\Gamma_t} \lambda(t) \; < \partial_t \varphi, n_t > \; d\Gamma_t dt + \int_0^\tau \int_{\Gamma_t} \lambda(t) < -D\varphi.V + DV.\varphi, n_t > \; d\Gamma_t dt
\end{aligned}$$

The first term $E_1 = -\displaystyle\int_0^\tau \int_{\Gamma_t} \lambda \; (\partial_t \varphi).n_t \; d\Gamma_t dt$ is treated as follows : Using the transformation $T_t(V)$

$$\begin{aligned}
E_1 &= -\int_0^\tau \int_\Gamma \lambda \circ T_t \; [(\partial_t \varphi) \circ T_t] \,.\, n_t \circ T_t \,\omega(t)\; d\Gamma dt, \;\; \omega(t) = det(DT_t) \parallel {}^*DT_t^{-1}.n \parallel_{R^N} \\
&= -\int_0^\tau \int_\Gamma < \partial_t(\varphi \circ T_t) - (D\varphi.V) \circ T_t \,, \; n_t \circ T_t > \; \lambda \circ T_t \,\omega(t)\; d\Gamma dt \\
&= \int_0^\tau \int_\Gamma < \varphi \circ T_t \,, \, n_t \circ T_t > \; \partial_t(\lambda \circ T_t)\,\omega(t) + \; < \varphi \circ T_t \,, \, \partial_t(\omega(t)\, n_t \circ T_t) > \; \lambda \circ T_t \, d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma < (D\varphi.V) \circ T_t \,, \, n_t \circ T_t > \; \lambda \circ T_t \,\omega(t)\; d\Gamma dt.
\end{aligned}$$

But $\omega(t) n_t \circ T_t = \gamma(t)\, {}^*DT_t^{-1} n\,$; where $\gamma(t) = \det DT_t$ so

$$
\begin{aligned}
E_1 =& \int_0^\tau \int_\Gamma \partial_t(\lambda \circ T_t) < \varphi \circ T_t\,,\, n_t \circ T_t > \,\omega(t)\, d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma < \varphi \circ T_t\,,\, \partial_t(\gamma(t)\, {}^*DT_t^{-1})n) > \,\lambda \circ T_t\, d\Gamma dt, \\
&+ \int_0^\tau \int_{\Gamma_t} \lambda < D\varphi.V\,,\, n_t > \, d\Gamma_t dt, \\
=& \int_0^\tau \int_\Gamma [\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V] \circ T_t < \omega(t)\, n_t \circ T_t\,, \varphi \circ T_t > \, d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma \lambda \circ T_t < \varphi \circ T_t\,,\, \partial_t(\,{}^*DT_t^{-1})n > \,\gamma(t)\, d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma \lambda \circ T_t < \varphi \circ T_t\,,\, \partial_t(\gamma(t))\,\, {}^*DT_t^{-1} n > \, d\Gamma dt + \int_0^\tau \int_{\Gamma_t} \lambda < D\varphi.V\,,\, n_t > \, d\Gamma_t dt \\
=& \int_0^\tau \int_{\Gamma_t} [\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V]\, \varphi.n_t\, d\Gamma_t dt + \int_0^\tau \int_{\Gamma_t} < D\varphi.V\,,\, n_t > \,\lambda\, d\Gamma_t dt \\
&+ \int_0^\tau \int_\Gamma \lambda \circ T_t < \varphi \circ T_t\,,\, n_t \circ T_t > \,\omega(t)\, (\mathrm{div} V) \circ T_t\, d\Gamma dt \\
&- \int_0^\tau \int_\Gamma \lambda \circ T_t < \varphi \circ T_t\,,\, {}^*(DV) \circ T_t\, n_t \circ T_t > \,\omega(t)\, d\Gamma dt \\
=& \int_0^\tau \int_{\Gamma_t} (\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V)\, \varphi.n_t\, d\Gamma_t dt + \int_0^\tau \int_{\Gamma_t} \lambda < D\varphi.V\,,\, n_t > \, d\Gamma_t dt \\
&+ \int_0^\tau \int_{\Gamma_t} \lambda\, \varphi.n_t\, \mathrm{div} V\, d\Gamma_t dt - \int_0^\tau \int_{\Gamma_t} \lambda < DV.\varphi\,,\, n_t > \, d\Gamma_t dt.
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
&< -\partial_t \mathbf{X} - D\mathbf{X}.V - \,{}^*DV.\mathbf{X} - \mathrm{div} V\, \mathbf{X}\,,\, \varphi_{\mathcal{D}',\mathcal{D}} > \\
&= \int_0^\tau \int_{\Gamma_t} (\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V + \lambda \mathrm{div} V)\, \varphi.n_t\, d\Gamma_t dt = \int_0^\tau \int_{\Gamma_t} f(t)\, \varphi(t).n_t\, d\Gamma_t dt
\end{aligned}
$$

which is equivalent, in a distribution sense, to

$$
-\partial_t \mathbf{X} - D\mathbf{X}.V - \,{}^*DV.\mathbf{X} - \mathrm{div} V\, \mathbf{X} = \gamma_{\Gamma_t}^*(f(t) n_t)
$$

Moreover it is clear that $\mathbf{X}(\tau) = 0$.
From the uniqueness theorem 2, we deduce that $\Lambda(t) = -\gamma_{\Gamma_t}^*(\lambda(t) n_t)$.

□

3 Derivability with respect to the field

As mentioned before, we are interested in the structure of the Eulerian derivative of non-cylindrical functionals of the following type

$$\left\{ \begin{array}{l} j(V) = \displaystyle\int_{Q(V)} F(t,x,u(V)(t,x))\, dxdt \\ \\ \text{where } \; Q(V) = \bigcup_{0<t<\tau} (\{t\} \times T_t(V)(\Omega)) \end{array} \right.$$

τ is a non-negative scalar.
Ω a domain in R^N.
$F : I(= [0,\tau]) \times D \times R^{N'} \to R \quad \mathcal{C}^1$.
The function u is solution of a well-posed non-cylindrical PDE, of order $2m$, $m \in N^*$, in $Q(V)$:

$$\partial_t u + A(u) = f \quad \text{in } Q(V) \tag{19}$$
$$B_j(u) = g_j \quad \text{on } \Sigma(V), \;\; 0 \le j \le m-1 \tag{20}$$
$$u(0) = u_0 \quad \text{in } \Omega \tag{21}$$

A is a differential operator of order 2m.
B_j is a boundary differential operator of order m_j $(0 \le m_j \le 2m-1)$.

In the sequel, the following notations will be used.

Notations 1
Assume the existence of the derivative, at $s = 0$, of the mapping $s \to u^s(.,.) = u(V + sW)(., T_s(\mathcal{Z}(s)(.))$ in $L^2(I, H^{2m}(\Omega_t(V))$ for the weak or the strong topology. It is denoted

$$\dot{u}(V;W).$$

Under the same assumption, let

$$u'(V;W) = \dot{u}(V;W) - \partial_x u.\mathbf{Z}.$$

Lemme 6
Assume that, for any direction $W \in \mathcal{C}([0,\tau], \mathcal{D}(D, R^N))$, the derivative $\dot{u}(V;W)$ exists in $L^2(I, H^{2m}(\Omega_t(V)))$ and that

$$u'(V;W) \quad \text{depends linearly on} \quad W.$$

Then the functional $j(.)$ is Gâteaux differentiable at V and there exists a time-dependent distribution $G(V) \in L^1(0,\tau;\mathcal{D}'(D,R^N))$ with spt$[G(V)(t)] \subset \overline{\Omega_t}(V)$ such that

$$j'(V;W) = \int_0^\tau < G(V)(t), W(t) >_{\mathcal{D}'(D,R^N),\mathcal{D}(D,R^N)} \; dt.$$

Proof. In the perturbed tube $Q(V+sW)$, the cost functional has the following expression :

$$\begin{aligned} j(V+sW) &= \int_0^\tau \int_{\Omega_t(V+sW)} F(t,x,u(V+sW)(t,x))\,dxdt \\ &= \int_0^\tau \int_{T_s(\mathcal{Z}^t)(\Omega_t(V))} F(t,x,u(V+sW)(t,x))\,dxdt \\ &= \int_0^\tau \int_{\Omega_t(V)} F(t,T_s(\mathcal{Z}^t)(x),u^s(t,x))\,dxdt \end{aligned}$$

Hypothesis *iii*) ensures the existence of

$$\mathbf{j}'(\mathbf{V};\mathbf{W}) = \frac{\mathbf{d}}{\mathbf{ds}}\mathbf{j}(\mathbf{V}+\mathbf{sW})_{|\mathbf{s=0}}.$$

Precisely we have

$$\begin{aligned} j'(V;W) &= \int_0^\tau \int_{\Omega_t(V)} \partial_x F(t,x,u(t,x)).\mathbf{Z}(t,x) + \partial_y F(t,x,u(t,x)).\dot{u}(t,x) \\ &+ F(t,x,u(t,x))\mathrm{div}\mathbf{Z}(t,x)\,dxdt \end{aligned}$$

which is equivalent to

$$\begin{aligned} j'(V;W) &= \int_0^\tau < \partial_y F(t,x,u(t,x)); \dot{u}(t,x) - \partial_x u(t,x).\mathbf{Z}(t,x) > \,dt \\ &+ \int_0^\tau \int_{\Omega_t(V)} \mathrm{div}[F(t,x,u(t,x))\mathbf{Z}(t)]\,dxdt \end{aligned}$$

which we rewrite, under smoothness assumptions, as follows

$$j'(V;W) = \int_0^\tau \int_{\Omega_t(V)} \partial_y F(t,x,u(t,x)).u'(t,x)\,dxdt \tag{22}$$
$$+ \int_0^\tau \int_{\Gamma_t(V)} F(t,x,u(t,x))\mathbf{Z}(t).n_t\,d\Gamma_t dt.$$

According to assumption ii) and the linear dependence of $\mathbf{Z}$ on W, we obtain the linear dependence of $\mathbf{j}'(\mathbf{V};\mathbf{W})$ on W.

□

Considering Λ the solution of problem (15)-(16) we can express the boundary integral on $\mathbf{Z}(t).n_t$ explicitly in terms of $W(t).n_t$. It is the object of the following lemma.

Lemme 7
Let F be a sufficiently smooth function defined on $\Sigma(V)$. Then

$$\int_0^\tau \int_{\Gamma_t(V)} F(t)\,\mathbf{Z}(t).n(t)\,d\Gamma_t dt =$$

$$\int_0^\tau \int_{\Gamma_t(V)} \left\{ \int_t^\tau F(s) \circ T_s(V) \circ T_t(V)^{-1} \, ds \right\} W(t).n(t) \, d\Gamma_t dt$$

Proof.

$$\int_0^\tau \int_{\Gamma_t} F(t) \, \mathbf{Z}.n_t \, d\Gamma_t dt = \int_0^\tau < -\partial_t \Lambda - D\Lambda.V - {}^*DV.\Lambda, \mathbf{Z} > \, dt$$

$$= \int_0^\tau < \partial_t \mathbf{Z} + D\mathbf{Z}.V - DV.\mathbf{Z}, \Lambda > \, dt = -\int_0^\tau \int_{\Gamma_t(V)} \lambda(t) \, W.n_t \, d\Gamma_t dt$$

$$= \int_0^\tau \int_{\Gamma_t} \int_t^\tau F(s) \circ T_s \circ T_t(V)^{-1} \, W(t).n(t) [\gamma(s)\gamma(t)] \circ T_t(V)^{-1} \, ds \, d\Gamma_t dt$$

□

Remark 3
In fact, we can use the explicite expression of $\mathbf{Z}$ given by (11) to obtain the expression of the integral in terms of W and by-pass the adjoint problem associated to $\mathbf{Z}$.

In the sequel, it will be shown that the eulerian derivative coincides with the shape derivative when the functional depends only on the shape of the tube. First let us define a *tube function* (resp. *tube functional*).

Lemme 8
If $u(V+W) = u(V)$ (*resp.* $\mathbf{j}(\mathbf{V}+\mathbf{W}) = \mathbf{j}(\mathbf{V})$) , for any sufficiently smooth V and W s.t. $W(t).n_{\Omega_t(V)} = 0$ on $\Sigma(V)$, then u (*resp.* $\mathbf{j}$) depends only on the shape of the considered tube. It is called a *tube function* (resp. *tube functional*).

Proposition 5
The hypotheses of Lemma 6 are assumed to be satisfied.

1. If u depends only on the trace, on the lateral boundary $\Sigma(V)$, of the field V, then the gradient $G(V)$ is supported on $\Sigma(V)$ and there exists $R(V) \in L^1(0,\tau;\mathcal{D}'(\Gamma_t(V))$ s.t.
$$G(V)(t) = \gamma^*_{\Gamma_t(V)}(R(V)(t))$$
where $\gamma^*_{\Gamma_t(V)}$ is the adjoint of the trace operator on Γ_t.
2. If u is a tube function (so it is denoted $u(Q_V)$), then
$$j'(V;W) = 0 \quad \text{for any } W \;\; s.t. \;\; W(t).n_{\Omega_t(V)} = 0 \;\; in \;\; \Gamma_t(V) \;\; for \; a.e. \;\; t \in [0,\tau).$$
3. Assume Ω of class $\mathcal{C}^k$, $k \geq 1$ and the linear mapping $W \longmapsto \mathbf{j}(\mathbf{V};\mathbf{W})$ continuous in $\mathcal{C}([0,\tau]; C^k(\overline{D}, R^N))$. Under the assumptions of Lemma 6 and if u is a tube function, there exists a time-dependent distribution g, $g(t) \in [\mathcal{D}^{k-1}(\Gamma_t(V))]'$, such that
$$\mathbf{j}'(\mathbf{V};\mathbf{W}) = \int_{\mathbf{0}}^\tau < \gamma^*_{\mathbf{\Gamma_t(V)}}(\mathbf{g}(\mathbf{V})(\mathbf{t})\mathbf{n_t}), \mathbf{W}(\mathbf{t}) >_{\mathcal{D}^{\mathbf{k-1}}(\mathbf{D})', \mathcal{D}^{\mathbf{k-1}}(\mathbf{D})} \; \mathbf{dt}$$

Proof.

1. Let W be a field such that $W(t) \in \mathcal{D}(D, R^N)$ and $\mathrm{spt}W(t) \cap \overline{\Omega_t(V)} = \emptyset$ for any $t \in [0, \tau)$. Thus $T_t(V+sW)\Omega = T_t(V)\Omega\ (= \Omega_t(V))$. Therefore $Q(V+sW) = Q(V)$. On the other-hand, $u(V+sW)(t)_{|\Gamma_t(V)} = u(V)(t)_{|\Gamma_t(V)}$ a.e. in $[0, \tau]$. The well-posedness of the PDE satisfied by u implies its uniqueness. So, in this case, we have $u(V+sW) = u(V)$ a.e. in $Q(V)$. This proves that $\mathbf{j}'(\mathbf{V}; \mathbf{W}) = \mathbf{0}$. So $\mathrm{spt}G(V)(t) \subset \overline{\Omega_t(V)}$ a.e. in $[0, \tau]$.
   By similar arguments and considering vector fields W such that $W(t) \in \mathcal{D}(\Omega_t(V), R^N)$ for any $t \in [0, \tau)$, we prove that $\mathrm{spt}G(V)(t) \subset \overline{\Omega_t(V)^c}$ for any $t \in [0, \tau)$. Hence, we can conclude that $\mathrm{spt}G(V)(.) \subset \Sigma(V)$.
2. Expressing the functional $\mathbf{j}$ at the "point" $V + sW$, we obtain
   $$j(V+sW) = J(V+sW, Q(V+sW)) = J\left(V+sW, \bigcup_{0<t<\tau} [\{t\} \times \mathcal{T}_s^t \Omega_t(V)]\right)$$
   where $\mathcal{T}_s^t = T_t(V+sW) \circ T_t(V)^{-1}$.
   The condition $W(t).n_{\Omega_t(V)} = 0$ for any $t \in [0, \tau)$ implies that $T_t(V+sW)\Omega = T_t(V)\Omega\ (= \Omega_t(V))$. It comes that $Q(V+sW) = Q(V)$. Moreover $u(V+sW) = u(V)$, then $\mathbf{j}(V+sW) = \mathbf{j}(\mathbf{V})$. Therefore $\mathbf{j}'(\mathbf{V}; \mathbf{W}) = \mathbf{0}$.
3. The continuity of the mapping $W \longmapsto \mathbf{j}(\mathbf{V}; \mathbf{W})$ in $\mathcal{C}([0, \tau]; C^k(\overline{D}, R^N))$ and the fact that the gradient $G(V)(t) = \gamma^*_{\Gamma_t(V)}(R(V)(t))$, give that $R(V) \in L^1(0, \tau; \mathcal{D}^{-k+1}(\Gamma_t(V)))$. Moreover
   $$j'(V; W) = \int_0^\tau < R(V)(t), W(t) >_{\mathcal{D}^{-k+1}, \mathcal{D}^{k-1}}\ dt$$
   $$= \int_0^\tau < R(V)(t), C(t) >\ dt + \int_0^\tau < R(V)(t), (W(t).n_t(V))n_t(V) >\ dt$$
   $$= \int_0^\tau < R(V)(t), (W(t).n_t(V))\, n_t(V) >\ dt$$
   where C is an admissible vector field such that
   $$C(t) = W(t) - (W(t).n_t(V))\, n_t(V) \ \text{ on } \Gamma_t(V).$$
   We know that $j'(V; C) = 0$. So under the specified hypotheses
   $$R(V)(t) = \gamma^*_{n_t}(g(V)(t) n_t(V)).$$
   where γ_{n_t} is the normal trace.

□

Remark 4

1. More generally if u depends only on the trace, on the lateral boundary $\Sigma(V)$, of the field V and $R(V) \in L^1(0, \tau; \mathcal{D}^{-k+1}(\Gamma_t(V))$ then

$$j'(V; W) =$$

$$\int_0^\tau < R(V)(t), (W(t).n_t(V))\, n_t(V) > \,dt + \int_0^\tau < R(V)(t), C(t) > \,dt$$

where the first integral of the right-hand term is the shape derivative and the second one is purely dynamic and is due to the variation of the tangential component of W.

2. Assume the existence of a vector function R with $R(t) \in L^p(\Gamma_t(V), R^N)$, $p \geq 1$, such that $G(V)(t) = \gamma^*_{\Gamma_t(V)} R(t)$. Then, under the assumptions of Lemma 6 and if the density g satisfies $g(t) = R(t).n_t \in L^p(\Gamma_t(V))$, we have an integral representation for the derivative

$$j'(V; W) = \int_0^\tau \int_{\Gamma_t(V)} g(V)(t)\, W(t).n_t \, d\Gamma_t \, dt.$$

4 Newton-Shape Method

We apply the previous tube analysis to a usual shape functional $J(\Omega)$. For a given field V, the domain $\Omega_t(V)$ is defined for any t then we get the classical expension (25) that we treat following the previous tube functional approach. The "initial" domain Ω being given as well as the time t_0, from (25) the functional $J(\Omega_{t_0}(V))$ turns to be a functional $j(V)$ of the speed vector field V, $j(V) = J(\Omega_{t_0}(V))$ being defined by the right hand side of (25). Then we apply our previous tube derivative calculus with the use of the transverse field Z associated to the perturbation field W. Concerning the shape analysis we adopt now the classical terminologie introduced in [8], [12] concerning the notions of shape derivative(resp. boundary shape derivative) of distributions defined on a moving domain (resp. on a moving boundary or surface). For example the boundary shape derivative of the normal vector field, denoted by $n'_\Gamma(V)$ is given by $n'_\Gamma(V) = -\nabla_\Gamma(< V(0), n >)$ see [5].

Assume the family of domains to be smooth enough, say Ω of class C^k with the integer $k \geq 1$. We know that for any

$$V \in \mathcal{E}_{0,k} \overset{def}{=} C^0([0, \tau], \mathcal{V}^k_o(D))$$

the associated flowand its inverse , $T(V)$ and $T(V)^{-1}$, are elements of $\mathcal{C}^1([0, \tau], \mathcal{C}^k(\overline{D}, R^N))$. Consider a shape functional $J(.)$ defined on a such family of domains $\mathcal{A}$ containing the domains

$$\{\Omega_s(Y) = T_s(Y)\Omega; \forall s \in [0, \tau], \forall Y \in \mathcal{V}^k_o(D)\}.$$

Suppose J shape differentiable on any $B \in \mathcal{A}$. We denote by $dJ(B; Y)$ the shape derivative of J on any $B \in \mathcal{A}$ in the direction Y. If B is in $\mathcal{C}^k(\overline{D}, R^N)$, there exists a Distribution

$G(B)$ of finite order, $G(B) \in \mathcal{D}^{k-1}(D, \mathbb{R}^n)'$ such that

$$dJ(B;Y) =< G(B), Y >_{\mathcal{D}^{k-1}(D,\mathbb{R}^n)', \mathcal{D}^{k-1}(D,\mathbb{R}^n)} .$$

When $k = 1$ that gradient is a (vector) measure supported by the boundary.

4.1 expension

We assume that the gradient is smooth enough in time, ($s \to G(\Omega_s(V)) \in L^1(0, \tau, \mathcal{D}'(D, R^N))$, then we obtain the following expension :

$$j(V) = J(\Omega_{t_0}(V)) = J(\Omega) + \int_0^{t_0} < G(\Omega_s(V)) \, , \, V(s) > \, ds \tag{23}$$

We say that the shape functional J is twice shape differentiable (see [10]) if

- The mapping $\Omega \to G(\Omega)$ is shape differentiable, that is :
  $\forall V$, the mapping
  $$\begin{array}{ccc} R & \longrightarrow & \mathcal{D}^{k-1}(D, R^N)' \\ s & \longmapsto & G(\Omega_s(V)) \end{array}$$
  is derivable. The shape derivative, which is the derivative at $s = 0$ is denoted $G'(\Omega; V)$.
  We assume that the mapping $Z \longrightarrow G'(\Omega_s(V), Z)$ is continuous and then (see [7], [8]) depends only on $V(s)$ then we get $G'(\Omega_s(V), Z) = G'(\Omega_s(V), Z(s))$.
  Moreover we assume that the mapping $Z \to G'(\Omega_s(V), Z)$ is linear and continuous and since $\partial\Omega_s(V)$ is smooth, from the generic argument developped in [10] we derive that $G'(\Omega, .)$ will depend on the autonomeous field V only through its normal componant on the boundary, that is $G'(\Omega_s(V)).Z = \tilde{G}'(\Omega_s(V)).(z(s))$ where $z(s) =< Z(s), n_s >$.
  We define the associated *Shape Hessian* (which can be seen as a continuous linear operator) $\tilde{G}'(\Omega_s(V)) \in \mathcal{L}(\mathcal{D}(\Gamma, R^N), \mathcal{D}(D, R^N)')$.
- We suppose the mapping $s \longmapsto G'(\Omega_s(V))$, to be continuous.
  The derivative of j on V in the direction W is
  $$j'(V;W) = \int_0^{t_0} < G'(\Omega_s(V)) \cdot \mathbf{Z} \, , \, V > + < G(\Omega_s(V)), W > \, ds$$

From [9] the structure theorem ([9] [8], [4]) for gradient, there existe a scalar distribution, called shape density gradient, $g(B)$ in $\mathcal{D}^{k-1}(\partial B)'$ such that

$$dJ(B;Y) =< g(B); \gamma_{\partial B}(Y).n >_{\mathcal{D}^{k-1}(\Gamma)', \mathcal{D}^{k-1}(\Gamma)} \tag{24}$$

We introduce the class of Shape differentiable functionals as follows :

Definition 1
The shape functional J is in $\mathcal{H}^1(\mathcal{A})$ if it is shape differentiable at any $B \in \mathcal{A}$ and if the density gradient $g(B)$ of J at $B \in \mathcal{A}$ verifies : $g(B) \in H^{\frac{1}{2}}(\partial B)$.

Assume J in $\mathcal{H}^2(\mathcal{A})$ and consider the following expension :

$$J(\Omega_{t_0}(V)) = J(\Omega) + \int_0^{t_0} < G(\Omega_s(V))\,,\, V(s) > \, ds \tag{25}$$

We are interested in computing the first order optimality condition for the minimization problem

$$\min_V J(\Omega_{t_0}(V)), \quad t_0 \text{ being fixed.}$$

We suppose that, $\forall V$, the mapping $s \longmapsto \tilde{G}'(\Omega_s(V))$, to be continuous.(That is equivalent to say that the mapping $\Omega \to G'(\Omega)$ is continuous fom $\mathcal{A}$ equipped with the *Courant Metric of domains* (see [4]) in $\mathcal{D}(D, R^N)$.

4.2 Second derivative with density gradient formulation

The derivative of j on V in the direction W is

$$j'(V;W) = \int_0^{t_0} < G'(\Omega_s(V)) \cdot \mathbf{Z}(s)\,,\, V > + < G(\Omega_s(V)), W > \, ds$$

If we rewrite j in terms of the density gradient $g(\Gamma_s(V))$ associated to $G(\Omega_s(V))$, using the fact that

$$G(\Omega_s(V)) = \gamma^*_{\Gamma_s(V)}(g(\Gamma_s(V))\, n(\Gamma_s(V))) \tag{26}$$

So that

$$j(V) = J(\Omega) + \int_0^{t_0} \int_{\Gamma_s(V)} g(\Gamma_s(V)) \; < V(s), n(\Gamma_s(V)) > \, d\Gamma_s ds$$

thus the derivative of j at a point V, in the direction W, has a more explicit expression.

Proposition 6
Assume

1. the data sufficiently smooth
2. the shape derivative $g'(\Gamma_s(V); Y)$ exists for any admissible autonomous direction Y and any $s \in [0, t_0]$.

Then $\Gamma_s(V)$,

$$\begin{aligned} j'(V;W) = \int_0^{t_0} \int_{\Gamma_s(V)} & \big(\; g'(\Gamma_s(V); \mathbf{Z}(s)) \; < V(s), n(\Gamma_s(V)) > + \\ & g(\Gamma_s(V)) \; < W(s), n(\Gamma_s(V)) > + \, div_{\Gamma_s(V)}(\, g(\Gamma_s(V))\, V(s)) \; < \mathbf{Z}(s),\, n_s > + \\ & < \nabla_{\Gamma_s}(\; g(\Gamma_s(V))\;),\, V_{\Gamma_s(V)}(s) \;> < \mathbf{Z}(s),\, n_s > \\ & + g(\Gamma_s(V)) \; < DV(s).n_s,\, n_s \;> \; < Z(s),\; n_s \;> \quad \big)\, d\Gamma_s ds \end{aligned}$$

and it depends on $W(s)$ through $Z(s)$.

Proof. By direct calculation, following derivativés rules and notations ([8], [5]),, κ being the mean curvature of the manifold, we get :

$$j'(V;W) = \int_0^{t_0} \int_{\Gamma_s(V)} \Big(\; g'(\Gamma_s(V);\mathbf{Z}(s)) \; < V(s), n(\Gamma_s(V)) > +$$
$$g(\Gamma_s(V)) \; < W(s), n(\Gamma_s(V)) > + g(\Gamma_s(V)) \; < V(s), n'(\Gamma_s(v); Z(s)) > +$$
$$\kappa\, g(\Gamma_s(V)) \; < V(s), n(\Gamma_s(V)) >< \mathbf{Z}(s), n(\Gamma_s(V)) >$$
$$+ \; < DV(s).n_s, \; n_s \; > \; < Z(s), \; n_s \; > \quad \Big) \, d\Gamma_s ds$$

Now as $n'(\Gamma_s(v); Z(s)) = -\nabla_{\Gamma_s(v)}(< Z(s), n_s >)$, using tangential by part integration we obtain (27). □

We assume the following linearity : $g'(\Gamma_s(V); \mathbf{Z}(s)) = \tilde{g}'(\Gamma_s(V)). < \mathbf{Z}(s)), \, n_s >$, which is true from the structure derivative theorem ([7], [8]), as soon as $g'(\Gamma_s(V); \mathbf{Z}(s))$ is linear and continuous with respect to Z, together with the regularity of the boundary. The first term can be rewritten as :

$$\int_0^{t_0} \int_{\Gamma_s(V)} (\, g'(\Gamma_s(V); \mathbf{Z}(s)) \; < V(s), n(\Gamma_s(V)) > \, d\Gamma_s ds$$

$$= \int_0^{t_0} \int_{\Gamma_s(V)} (\, \tilde{g}'(\Gamma_s(V))^*. < V(s), n(\Gamma_s(V)) >< \mathbf{Z}(s), n(\Gamma_s(V)) > \, d\Gamma_s ds$$

Let

$$F(s) = \tilde{g}'(\Gamma_s(V))^*. < V(s), n(\Gamma_s(V)) > + g(\Gamma_s(V)) div V(s) + < \nabla_{\Gamma_s(V)} g(\Gamma_s(V)), \, V(s) >$$

Then

$$j'(V;W) = \int_0^{t_0} \int_{\Gamma_s(V)} F(s)\mathbf{Z}(s).n(s) + g(\Gamma_s(V))W(s).n(s) \quad d\Gamma ds$$

Finally we derive the result concerning the necessary Optimality condition for $j(V) = J(\Omega_{t_0}(V))$:

Proposition 7
The necessary optimality condition associated to considered minimization problem is :

$$a.e.s \text{ in } (0, t_0) \; \{ \int_s^{t_0} F(r) \circ T_r(V) \circ T_s(V)^{-1} \, dr\} + g(\Gamma_s(V)) = 0 \quad) a.e. \text{ in } \Gamma_s(V)$$

The proof is based on Lemma 7. We can also use the adjoint problem solution λ and derive explicit expression for the gradient of the functional j in order to derive a second order descent method for the shape functional J. From 23 we get

$$\int_0^{\tau} \int_{\Gamma_t} F(t)\, \mathbf{Z}.n_t \, d\Gamma_t dt = - \int_0^{\tau} \int_{\Gamma_t(V)} \lambda(t)\, W.n_t \; d\Gamma_t dt$$

So that :

$$j'(V;W) = \int_0^{t_0} \int_{\Gamma_s(V)} (\ \lambda(s) + g(\Gamma_s(V))\ )\ < W(s), n(s) > \quad d\Gamma ds$$

Where λ solves the backward problem :

$$\partial_t \lambda(t) + \nabla_{\Gamma_t}\lambda.V + \lambda\,\mathrm{div}V = F(t) \ \text{ on } \ \cup_t\,(\{t\}\times\Gamma_t), \quad \lambda(t_0) = 0 \ \text{ on } \ \Gamma_{t_0}$$

4.3 Example

Finally, let us consider the following functional and the associated minimization over V :

$$j(V) = J(\Omega) - \frac{1}{2}\int_0^{t_0}\int_{\Gamma_t(V)} (\frac{\partial y_t}{\partial n_t})^2 \ < V(t),\ n_t > \ d\Gamma dt$$

The eulerian derivative of j in the direction W is given by :

$$j'(V;W) = -\frac{1}{2}\int_0^{t_0}\int_{\Gamma_t(V)} (\,[2\,(\frac{\partial y_t}{\partial n_t})^2 \ < V(t), n_t > \ + \mathrm{div}\ (|\nabla y_t(V)|^2 V)\ \ ] < \mathbf{Z}, n_t >$$

$$+\,(\frac{\partial y_t}{\partial n_t})^2 \ < W, n_t >)\, d\Gamma dt$$

So the necessary optimality condition would be the following :

$$\int_t^{t_0} [2(\frac{\partial y_s}{\partial n_s})^2 \ < V(s), n_s > \ + \mathrm{div}\ (|\nabla y_s(V)|^2 V(s))\ ] \circ T_s(V) \circ T_t^{-1}(V)\ ds \ + \ (\frac{\partial y_t}{\partial n_t})^2 \ = 0$$

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ISSN 0249-6399